Spherical normal forms for resonant saddle points in $\mathbb{C}^2$

Bifurcation of Dynamical Systems and Numerics, Zagreb

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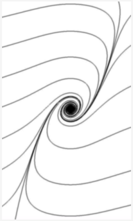
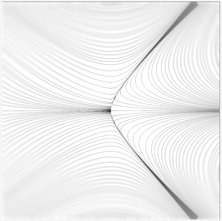
May 10^{th}, 2023

Context

Local holomorphic dynamical systems in the complex plane $\mathbb{C}^2$





Theory and actual computation / decision regarding:

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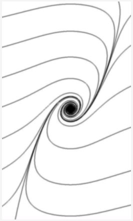
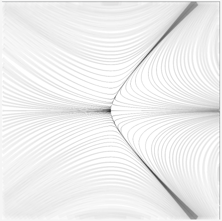

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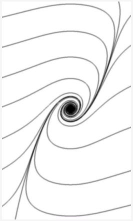

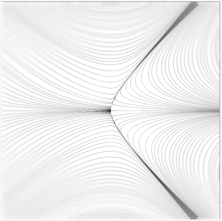

Theory and actual computation / decision regarding:

- normal forms of foliations $\mathcal{F}_X$ (=phase-portrait of vector field X)
- integrability of foliations $\mathcal{F}_X$ (=Liouvillian first-integral for X)

A bit of zoology

Reduced singularities

$$X(x,y) = (\lambda_1 x + \cdots)\frac{\partial}{\partial x} + (\lambda_2 y + \cdots)\frac{\partial}{\partial y} \quad , \quad \lambda_2 \neq 0$$

Eigenratio $\lambda := {}^{\lambda_1}/_{\lambda_2}$

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5. $\lambda \in \mathbb{R}_{<0} \backslash \mathbb{Q}$: quasi-resonant saddles, too complicated

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Compute simple unique forms (**normal forms**) for X up to $\Psi \in \mathrm{Diff}\left(\mathbb{C}^2, 0\right)$ and decide if X is integrable

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Actual aim

Do that for ~~reduced~~ resonant saddle singularities

What is known

Theorem (*Poincaré-Dulac 1904 / Bruno 1980*)

Formal normal forms for reduced, resonant singularities

$$\lambda = -{}^{p}/_{q} \text{ with } p \wedge q = 1 \text{ or } (p,q) = (0,1)$$

$$X \ \widehat{\sim} \ \widehat{X} := P(u)\left(xu^k \frac{\partial}{\partial x} + \left(1+\mu u^k\right)\left(-px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y}\right)\right)$$

(orbital) $k \in \mathbb{Z}_{>0},\ \mu \in \mathbb{C}$ ***(temporal)*** $P \in \mathbb{C}[u]_{\leq k}$

resonant monomial $u = u(x,y) := x^q y^p$

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Theorem (*Loray 2004 / Schäfke-Teyssier 2015*)

Analytic normal forms for saddle-nodes with 2 separatrices

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1. *Form unique up to* $GL_2(\mathbb{C})$

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Remark

It is possible to compute symbolically finite jets of the normal form, hence integrability is semi-decidable

What is (almost) known

Écalle 2005

There exists a universal family SNF (**spherical normal forms**), depending on a **twist parameter** and whose elements are obtained by summing *twisted resurgent monomials*, that is in correspondence with orbital analytic classes of resonant foliations

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Remark

It is not possible to extract from his work an explicit expression for SNF

What is not known

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- Quasi-resonant saddle points (Loray: simple forms but not unique nor computable)
- Saddles nodes with only 1 separatrix

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In the rest of the talk

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2. non-linearizable and most simple formal model

$$X \widehat{\sim} xu\frac{\partial}{\partial x} + \left(-x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \quad , \; u = xy$$

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Heuristics

analytic class of $\mathcal{F}_X$ = analytic class of leaf space of $\mathcal{F}_X$

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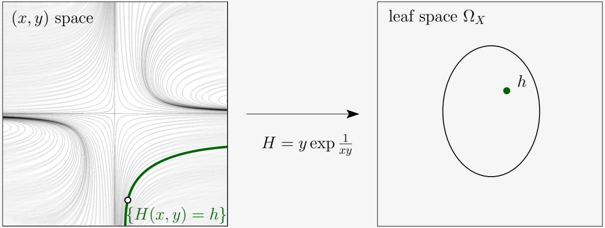

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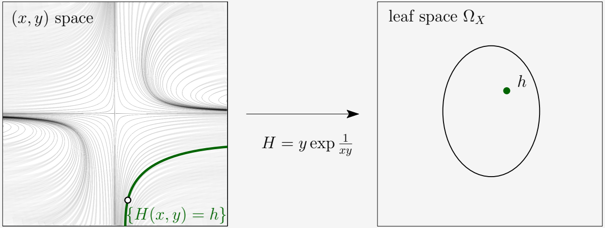

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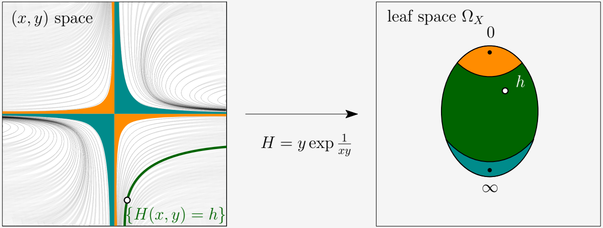
- $H\left(\mathbb{C}^2 \setminus \{u=0\}\right) = \mathbb{C}^\times$ and $\{u=0\}$ corresponds to $0,\ \infty$ (non-separable)



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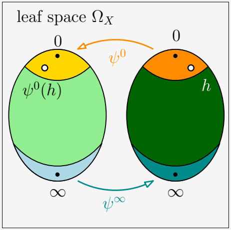

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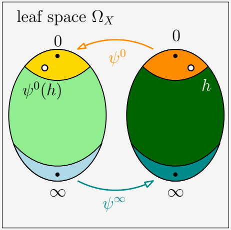
- a leaf crossing both sectors induces an identification in Ω_X
- that happens in neighborhoods of 0 and ∞



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Theorem (*Martinet-Ramis 1983*)

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3. Surjective: difficult $\rightarrow$**inverse problem**

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5. Newlander-Niremberg: $\mathcal{M} \simeq (\mathbb{C}^2, 0)$, and $\mathcal{F} \in \mathrm{FOL}$

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  - $\longrightarrow$ Constraint on the size of $\mathcal{V}^+ \cap \mathcal{V}^- \cap \left(\mathbb{C}^2, 0\right)$
- 'Newlander-Niremberg: $\mathcal{M} \simeq \left(\mathbb{C}^2, 0\right)$, and $\mathcal{F} \in \mathsf{FOL}$"
  - $\longrightarrow$ No control on the «shape» of $\mathcal{F}$

Inverse problem: «abstract» realization

Technical points

- "The manifold $\mathcal{M}$ is obtained by gluing $\mathcal{V}^+$ and $\mathcal{V}^-$ in h-space by $\left(\psi^0, \psi^\infty\right)$"
  - $\longrightarrow$ Need to control the size of $H\left(\mathcal{V}^+ \cap \mathcal{V}^-\right)$
  - $\longrightarrow$ Constraint on the size of $\mathcal{V}^+ \cap \mathcal{V}^- \cap \left(\mathbb{C}^2, 0\right)$
- 'Newlander-Niremberg: $\mathcal{M} \simeq \left(\mathbb{C}^2, 0\right)$, and $\mathcal{F} \in \mathrm{FOL}$"
  - $\longrightarrow$ No control on the «shape» of $\mathcal{F}$
  - $\longrightarrow$ No privileged choice (**normal form**)

Inverse problem: «concrete» realization

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- Introduce a **twist parameter** $c \gg 1$ to control $H(\mathcal{V}^+ \cap \mathcal{V}^-)$ and change the formal model:

$$X_0 := xu\frac{\partial}{\partial x} + c\left(1-u^2\right)\left(-x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$

$$H = y\exp\left(\frac{c}{u} + cu\right) \qquad \operatorname{diam}\left(H\left(\mathcal{V}^+ \cap \mathcal{V}^-\right)\right) = O\left(e^{-c}\right)$$

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- A holomorphic fixed-point allows to control the produced foliation

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1. Let $(\psi^0, \psi^\infty) \in \mathrm{Diff}(\mathbb{C},0)_{\mathrm{Id}} \times \mathrm{Diff}(\mathbb{C},0)_{\mathrm{Id}}$ be given. Choose c so that $H(\mathcal{V}^+ \cap \mathcal{V}^-) \subset \mathrm{domain}\left(\psi^{0,\infty}\right)$

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2. Synthesize by fixed-point $N^\pm$ bounded on $V^\pm \times \mathbb{C}$ so that

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3. Recover

$$R := \frac{xu \frac{\partial}{\partial x} N^\pm}{1 + \left(-x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) N^\pm} \in \mathbb{C}\{u, y\}$$

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4. Bounds on growth $\Longrightarrow R = uy(r_0(y) + ur_1(y))$ for $r_j \in \mathbb{C}\{y\}$

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1. *Form unique up to $GL_2(\mathbb{C})$*

$$X \sim \frac{1}{1+G}\left(xu\frac{\partial}{\partial x} + \left(c\left(1-u^2\right)+R\right)\left(-x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\right)$$

$$(\textbf{\textit{orbital}})\quad R \qquad (\textbf{\textit{temporal}})\quad G \qquad \in uy\mathbb{C}\{y\}[u]_{\leq 1}$$

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2. *Integrability $\Longleftrightarrow R \in y^d\mathbb{C}[u]_{\leq 1}$ (Bernoulli equation)*

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3. Triangular process, computable symbolically for all n

$$X_{\text{prepared}} = xu\frac{\partial}{\partial x} + (1 + uyA(x,y))\left(-x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$

$$X_R = xu\frac{\partial}{\partial x} + \left(c\left(1 - u^2\right) + uyR(u,y)\right)\left(-x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$

$$A = \sum_n a_n(x)\, y^n \longmapsto R = \sum_n (\alpha_n + \beta_n u)\, y^n$$

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$$\mathbb{C} \overset{cst}{\hookrightarrow} \mathbb{C}[[x,y]] \overset{X_R \cdot}{\longrightarrow} \mathbb{C}[[x,y]] \overset{\Pi}{\twoheadrightarrow} \mathbb{C}[u]_{\leq 1}$$

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Analytic action

$$\mathbb{C} \overset{cst}{\hookrightarrow} \mathbb{C}\{x, y\} \overset{X_R \cdot}{\longrightarrow} \ker \Pi \overset{\mathfrak{T}_R}{\twoheadrightarrow} h\mathbb{C}\{h\} \times \frac{1}{h}\mathbb{C}\left\{\frac{1}{h}\right\}$$

Period operator $\mathfrak{T}_R(f) = (\varphi^0, \varphi^\infty)$

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Proposition (Period operator and cohomological equations)

Any $f \in \mathbb{C}\{x, y\}$ *writes uniquely as*

$$f = P + G + X_R \cdot F \ , \quad \begin{cases} P & \in \mathbb{C}[u]_{\leq 1} \\ G & \in yu\mathbb{C}\{y\}[u]_{\leq 1} \\ F & \in \mathbb{C}\{x, y\} \end{cases}$$

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